Lecture 12
14.2/14.3 Continuity, partial derivatives

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Things to note

Collect Homework 04 (Questions?).

Last class

Theorem

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

$$\Leftrightarrow$$

$f(x,y)$ approaches the height L no matter what path approaching (a,b) in the domain is chosen.

Evaluating limits

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Determine if the following limit exists.

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$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right)$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{x - y}$$

$$= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) = 0$$

Properties of limits (page 803)

Theorem

Let L, M, and k be real numbers. Let

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \text{ and } \lim_{(x,y)\to(a,b)} g(x,y) = M.$$

Then the following hold.

1,2. $\lim_{(x,y)\to(a,b)}(f(x,y) \pm g(x,y)) = L \pm M$

3. $\lim_{(x,y)\to(a,b)} k(f(x,y)) = kL$

4. $\lim_{(x,y)\to(a,b)} f(x,y) \cdot g(x,y) = L \cdot M$

5. $\lim_{(x,y)\to(a,b)} f(x,y)/g(x,y) = L/M$ *if* $M \neq 0$

6. $\lim_{(x,y)\to(a,b)}[f(x,y)]^n = L^n$ *if* $n \in \mathbb{R}^+$

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Example

Functions like $e^{x+y}, \cos(\frac{xy}{x^2+1}), \ln(1+x^2y^2)$ are continuous on their domains, since they are compositions of continuous functions.

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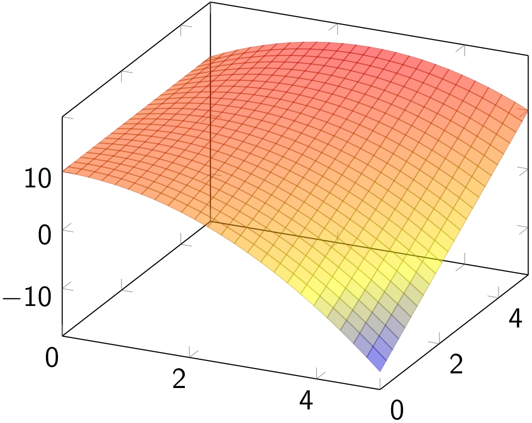

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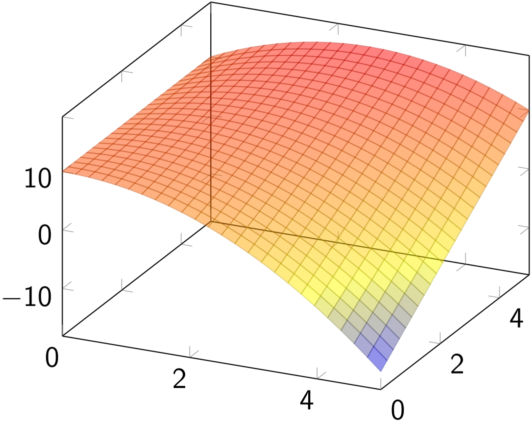

We can answer this question by thinking of $y = b$ as a constant, since we are only concerned with change in the x-direction.

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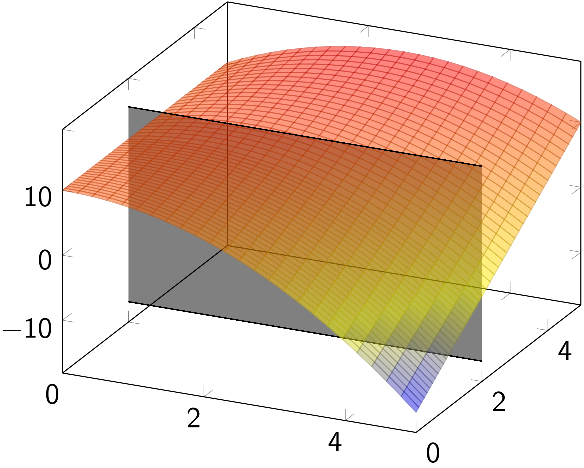
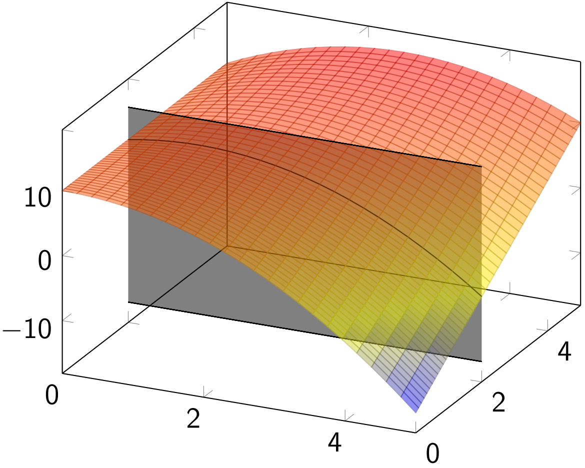


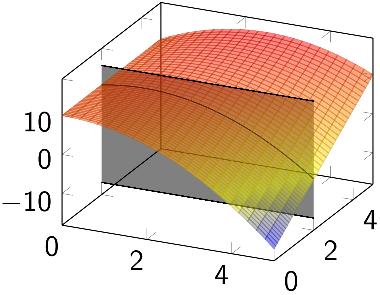
Partial derivatives

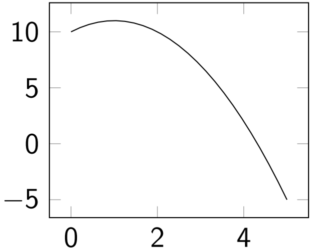
In the plane $y = b$, our function becomes $z = f(x, b)$, or just $z = f(x)$, a single variable function.



Partial derivatives

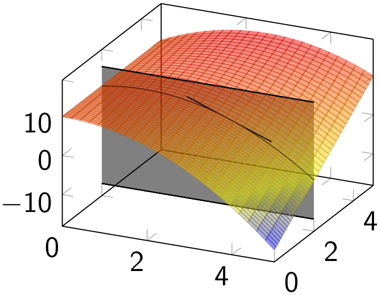
This reduces the picture to something we're familiar with from Calculus 1.

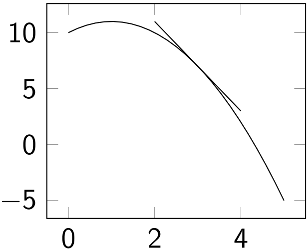




Partial derivatives

We can add in the tangent line by taking normal derivatives.





Partial derivatives

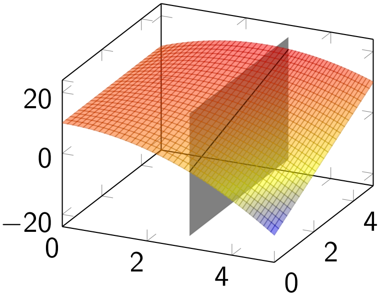
This process gives us an answer to our question:

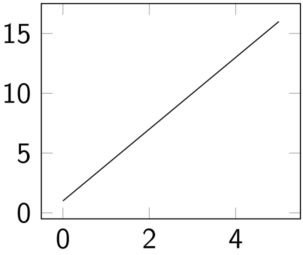
Answer

We find the slope of a function $z = f(x, y)$ *in the* x*-direction by treating* y *as a constant and differentiating with respect to* x*. The resulting function is a function that keeps track of the slope of* $f(x, y)$ *in the* x*-direction.*

Partial derivatives

We can do the same thing in the y-direction.





Formal definitions

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Definition

The partial derivative of $f(x, y)$ with respect to x is

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

The partial derivative with respect to y is

$$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

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However, in practice we will not use the definition and instead will use the various rules we learned in Calculus 1.

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Let $f(x, y) = x^2 + 3xy + y - 1$. *Find* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *at* $(4, -5)$.

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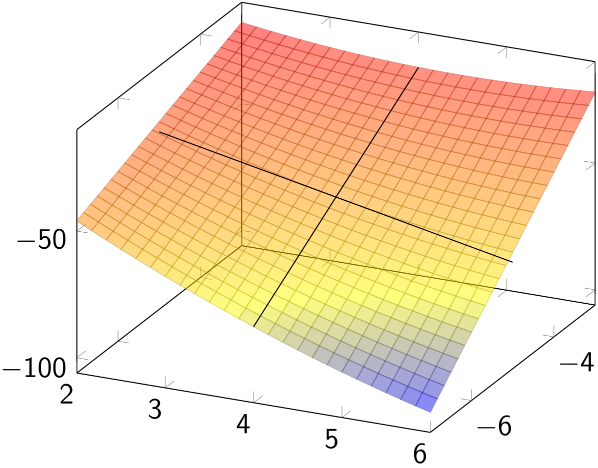
Treating y as a constant, we calculate $f_x(x,y) = \frac{\partial f}{\partial x} = 2x + 3y + 0$ (note that $\frac{\partial}{\partial x}\left[3xy\right] = 3y$ because the function is x times a constant rather than two functions of x multiplied together).

Similarly, we treat x as a constant to find $f_y(x,y) = \frac{\partial f}{\partial y} = 0 + 3x + 1$.

Evaluating these functions at $(4, -5)$, we find $f_x(4,-5) = 2(4) + 3(-5) = -7$ and $f_y(4,-5) = 3(4) + 1 = 13$. This means geometrically that the function is dropping at a slope of -7 in the x-direction and rising at a slope of 13 in the y-direction at the point $(4, -5)$.

Example picture

We get the following picture.



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Let $f(x, y) = y\sin(xy)$. *Find* $f_x(x, y)$ *and* $f_y(x, y)$.

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$f_x(x,y) = y\cos(xy)(y) = y^2\cos(xy)$.
$f_y(x,y) = (1)\sin(xy) + (y)(\cos(xy) * x) = \sin(xy) + xy\cos(xy)$.

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Definition

We say $f(x,y)$ *is differentiable at a point* (x_0, y_0) *in its domain if* f_x *and* f_y *are continuous near* (x_0, y_0)*.*

Second-order partial derivatives

We can partially differentiate a function more than once, and in multiple orders. There are four second-order partial derivatives.

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

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Let $f(x, y) = x\cos(y) + ye^x$. *Find all 2nd-order partial derivatives.*

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Let $f(x,y) = x\cos(y) + ye^x$. *Find all 2nd-order partial derivatives.*

$f_x = \cos(y) + ye^x$
$f_y = x(-\sin(y)) + e^x$
$f_{xx} = 0 + ye^x$
$f_{yy} = x(-\cos(y))$
$f_{yx} = (-\sin(y)) + e^x$
$f_{xy} = -\sin(y) + e^x$

Notice that $f_{yx} = f_{xy}$. This will be the case whenever $f(x,y)$ satisfies relatively lax criteria.

Mixed partials theorem

Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined near (a, b), then

$$f_{yx}(a, b) = f_{xy}(a, b).$$

This is known as Clairaut's Theorem.

In particular, if the conditions in the theorem hold for all the pairs (a, b) in the domain of the functions involved, then the functions will have the same formula on that domain, and we only need to find one of f_{xy} or f_{yx} to know the other.